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LETTER TO THE EDITOR

Linearization under non-point transformations

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Abstract. We find conditions for the linearization of a second-order differential equation under non-point transformations.

There has been some interest in the problem of determining when a given differential equation is equivalent to a linear differential equation [1–9]. The utilization of point transformations for this kind of linearization is the usual and most useful procedure. This transformation preserves the integrability of the equation and its Lie symmetry structure. For example, the most general second-order differential equation which is equivalent to the free particle equation

$$\mathrm{d}^2X/\mathrm{d}T^2 = 0 \tag{1}$$

has the form

$$\mathrm{d}^2x/\mathrm{d}t^2 + A_3(\mathrm{d}x/\mathrm{d}t)^3 + A_2(\mathrm{d}x/\mathrm{d}t)^2 + A_1\mathrm{d}x/\mathrm{d}t + A_0(x,t) = 0 \tag{2}$$

where the functions $A_i(x,t)$ satisfy the following conditions:

$$A_{1xx} - 2A_{2xt} + 3A_{3tt} + 6A_3A_{0x} + 3A_0A_{3x} - 3A_3A_{1t} - 3A_1A_{3t} - A_2A_{1x} + A_2A_{2t} = 0 \tag{3}$$

$$-A_{2tt} + 2A_{1xt} - 3A_{0xx} + 6A_0A_{3t} + 3A_3A_{0t} - 3A_0A_{2x} - 3A_2A_{0x} - A_1A_{2t} + 2A_1A_{1x} = 0. \tag{4}$$

These conditions were firstly deduced by Tresse [9]. The problem was also considered by Cartan [10] from a more geometrical point of view. The functions A_i are related to the invertible point transformations

$$\begin{aligned} X &= F(x,t) \\ T &= G(x,t) \end{aligned} \tag{5}$$

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by the equations

$$
\begin{aligned}
A_3 &= [G_x F_{xx} - G_{xx} F_x]/H \\
A_2 &= [G_t F_{xx} + 2G_x F_{tx} - 2F_x G_{xt} - F_t G_{xx}]/H \\
A_1 &= [G_x F_{tt} + 2G_t F_{tx} - 2F_t G_{tx} - F_x G_{tt}]/H \\
A_0 &= [G_t F_{tt} - G_{tt} F_t]/H
\end{aligned}
\tag{6}
$$

with $H = G_t F_x - G_x F_t \neq 0$. The Tresse–Cartan conditions (3) and (4) are the compatibility equations for the system (6) [5–9]. (There is a missing term in the expression (4–10) of [7]). When we know the point transformation (5) the knowledge of the invariants, symmetries and solutions of (1) can be directly used to obtain the corresponding ones of the equivalent equation. As any second-order linear differential equation is equivalent to the free particle equation, (3) and (4) are necessary and sufficient conditions for the linearization of the second-order differential equation. This result is not true for systems of second-order equations or for higher-order differential equations; in these cases the 'free particle' equation is not equivalent to a general linear equation under a point transformation [2, 10–12].

In spite of the utility of the point transformations, they are special and restrictive transformations in the sense that they preserve the Lie symmetry structure beyond the integrability of the equation. However, in several situations we want only to know if the equation is an integrable one and, eventually, to find integrals of motion and solutions. There are some types of non-point transformations that preserve the time-independent integrals of motion of the original equation and, consequently, permit us to find classes of integrable equations starting from an integrable one. We can find several specific situations, spread through the scientific literature, where these transformations are employed for identifying integrals of motion and solving differential equations [13–15]. We consider here the problem of finding the class of second-order equations that are equivalent under a non-point transformation (NPT) to the free particle equation (1).

We start with the NPT

$$
\begin{aligned}
X &= F(x, t) \\
\mathrm{d}T &= G(x, t)\, \mathrm{d}t
\end{aligned}
\tag{7}
$$

which is a generalization of the transformation originally proposed by Sundman (1912), and apply it to the equation (1). We get the following equation:

$$
\mathrm{d}^2 x/\mathrm{d}t^2 + A_2(\mathrm{d}x/\mathrm{d}t)^2 + A_1(\mathrm{d}x/\mathrm{d}t) + A_0 = 0 \tag{8}
$$

where

$$
\begin{aligned}
A_2 &= [G F_{xx} - F_x G_x]/K \\
A_1 &= [2G F_{tx} - F_t G_x - F_x G_t]/K \\
A_0 &= [G F_{tt} - F_t G_t]/K
\end{aligned}
\tag{9}
$$

with

$$
K = G F_x \neq 0.
$$

A procedure of systematic derivations of the equations (9), by using algebraic computation, permits us the elimination of the functions F, G and their derivatives and to find the compatibility conditions, analogous to the Tresse–Cartan conditions (3) and (4). These conditions lead to the following possibilities:

(i)

$$S_1(x,t) = A_{1x} - 2A_{2t} = 0 \tag{10}$$

$$S_2(x,t) = 2A_{0xx} - 2A_{1tx} + 2A_0A_{2x} - A_{1x}A_1 + 2A_{0x}A_2 + 2A_{2tt} = 0. \tag{11}$$

(ii) If $S_1(x,t) \neq 0$ and $S_2(x,t) \neq 0$ then

$$S_2^2 + 2S_{1t}S_2 - 2S_1^2A_{1t} + 4S_1^2A_{0x} + 4S_1^2A_0A_2 - 2S_1S_{2t} - S_1^2A_1^2 = 0 \tag{12}$$

and

$$S_{1x}S_2 + S_1^2A_{1x} - 2S_1^2A_{2t} - S_1S_{2x} = 0. \tag{13}$$

An invariant I for the equation (8), when the conditions (10) and (11) or (12) and (13) are satisfied, can be found directly from the invariant $I = \mathrm{d}X/\mathrm{d}T$ of equation (1):

$$I = \mathrm{d}X/\mathrm{d}T = (F_x/G)(\mathrm{d}x/\mathrm{d}t) + F_t/G \tag{14}$$

if we know the explicit form of the NPT (7). Here we take three examples.

(A) Consider the case (i) with $A_{1x} = 2A_{2t} = C$. We get, from (11), the condition

$$A_{0xx} + (A_0g)_x - (C/2)[Cx + h + A_0t] = 0 \tag{15}$$

and the equivalent equation

$$\mathrm{d}^2x/\mathrm{d}t^2 + [Ct/2 + g](\mathrm{d}x/\mathrm{d}t)^2 + [Cx + h](\mathrm{d}x/\mathrm{d}t) + A_0 = 0 \tag{16}$$

where $g = g(x)$ and $h = h(t)$ are arbitrary functions.

If $C = 0$, equation (16) can also be linearizable by a point transformation because it verifies, in this case, conditions (3) and (4). For example, the equation

$$\mathrm{d}^2x/\mathrm{d}t^2 - (2/x)(\mathrm{d}x/\mathrm{d}t)^2 + (2x/t^2) = 0 \tag{17}$$

can be linearizable by the point transformation

$$\begin{aligned} X &= xt^2 \\ T &= x/t \end{aligned} \tag{18}$$

or by the non-point transformation

$$\begin{aligned} X &= t^3x^{3/2} \\ \mathrm{d}T &= tx^{5/2}\,\mathrm{d}t. \end{aligned} \tag{19}$$

From the invariant $I = X - T(\mathrm{d}X/\mathrm{d}T)$ of (1) and from the point transformation (18), we get the following invariant for (17):

$$I_1 = 3x^2t^2/[t(\mathrm{d}x/\mathrm{d}t) - x]. \tag{20}$$

Another invariant for the same equation can be found, for instance, from the invariant $I = \mathrm{d}X/\mathrm{d}T$ and from the NPT (19):

$$I_2 = (t/x)[(t/2x)(\mathrm{d}x/\mathrm{d}t) + 1]. \tag{21}$$

If we start with the invariant $I = \mathrm{d}X/\mathrm{d}T$, the point transformation (18) leads to the invariant $I_3 = I_1 I_2/3$.

We observe that, in this case, unlike the general situation, the NPT (19) preserves the Lie symmetry structure of the original equation (1), as the point transformation always does.

(B) Now take the equation

$$\mathrm{d}^2x/\mathrm{d}t^2 + (t - 1/x)(\mathrm{d}x/\mathrm{d}t)^2 + 2x(\mathrm{d}x/\mathrm{d}t) + x^2/t - x/t^2 = 0. \tag{22}$$

It can be easily verified that (22) satisfies the conditions (12) and (13). Therefore, it is equivalent to (1). The NPT relating (1) and (22) has the form

$$\begin{aligned} F &= \mathrm{e}^{xt} \\ G &= xt. \end{aligned} \tag{23}$$

From (14) and (23), we find that (22) has the invariant

$$I = \mathrm{e}^{xt}[(\mathrm{d}x/\mathrm{d}t)(1/x) + 1/t]. \tag{24}$$

(C) If we start from a given NPT

$$\begin{aligned} F &= t\sin(x) \\ G &= xt \end{aligned} \tag{25}$$

for example, we get the following equation:

$$\mathrm{d}^2x/\mathrm{d}t^2 - [\tan(x) + 1/x](\mathrm{d}x/\mathrm{d}t)^2 + [1/t - \tan(x)/xt](\mathrm{d}x/\mathrm{d}t) - \tan(x)/t^2 = 0. \tag{26}$$

Of course, conditions (12) and (13) are verified in this case and, from (14), an invariant of (26) is

$$I = [(\mathrm{d}x/\mathrm{d}t)t\cos(x) + \sin(x)]/xt. \tag{27}$$

In spite of the complicated form of the conditions (10), (11) and (12) it is easy to check, with the help of a computer, if a given second-order differential equation can be reduced to (1) under NPT with the form (5). This method can be extended for getting the class of equations that are equivalent to a general linear equation. Of course, in the general case, it is difficult to solve the system of equations for finding the NPT joining two equivalent differential equations. But now we have a general algorithmic procedure for analysing the equivalence under NPT of given ordinary differential equations.

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